ON THE HOMOLOGY THEORY OF THE CLOSED GEODESIC PROBLEM

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Abstract. Let ΛX be the free loop space on a simply connected finite CW-complex X and $\beta_i(\Lambda X;\Bbbk)$ be the cardinality of a minimal generating set of $H^i(\Lambda X;\Bbbk)$ for $\Bbbk$ to be a commutative ring with unit. The sequence $\beta_i(\Lambda X;\Bbbk)$ grows unbounded if and only if $\tilde{H}^*(X;\Bbbk)$ requires at least two algebra generators. This in particular answers to a long standing problem whether a simply connected closed smooth manifold has infinitely many geometrically distinct closed geodesics in any Riemannian metric.

1. Introduction

Let Y be a topological space, let $\Bbbk$ be a commutative ring with unit, and assume that the i^{th}-cohomology group $H^i(Y;\Bbbk)$ of Y is finitely generated as a $\Bbbk$-module. We refer to the cardinality of a minimal generating set of $H^i(Y;\Bbbk)$, denoted by $\beta_i(Y;\Bbbk)$, as the *generalized i^{th}-Betti number of Y*. Let ΛX denote the free loop space, i.e., all continuous maps from the circle S^1 into X. In [5] Gromoll and Meyer proved the following

Theorem. *Let X be a simply connected closed smooth manifold of dimension greater than 1 and let $\Bbbk$ be a field of characteristic zero. If the Betti numbers $\beta_i(\Lambda X;\Bbbk)$ grow unbounded, then X has infinitely many geometrically distinct closed geodesics in any Riemannian metric.*

In fact, the proof of the theorem easily shows that the statement remains to be true for the Betti numbers $\beta_i(\Lambda X;\Bbbk)$ with respect to any field $\Bbbk$, too. Thus, this result has motivated a question, the 'closed geodesic problem', to find simple criteria which imply that the Betti numbers $\beta_i(\Lambda X;\Bbbk)$ are unbounded. Below we state such criteria in its most general form in the following

Theorem 1. *Let X be a simply connected space and $\Bbbk$ a commutative ring with unit. If $H^*(X;\Bbbk)$ is finitely generated as a $\Bbbk$-module and $H^*(\Lambda X;\Bbbk)$ has finite type, then the generalized Betti numbers $\beta_i(\Lambda X;\Bbbk)$ grow unbounded if and only if $\tilde{H}^*(X;\Bbbk)$ requires at least two algebra generators.*

Theorem 1 was proved by Sullivan and Vigué-Poirrier [20] over fields of characteristic zero, and then it was conjectured for $\Bbbk$ to be a field of positive characteristic. A number of papers [22], [19], [11], [12], [13], [15], [6], [14] deals with this conjecture but it remained to be open for X to be a finite CW-complex and $\Bbbk$ a finite field.

Here we prove Theorem 1. More precisely, it is a consequence of the following more general algebraic fact: Let $A=\{A^i\}, i\in\mathbb{Z}$, be a torsion free graded abelian

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group such that A is an associative Hirsch algebra and the bar construction BA is a Hirsch algebra [16]; this in particular means that A is an (associative) graded differential algebra (dga) endowed with higher order operations $E=\{E_{p,q}\}$ and $E'=\{E'_{p',q'}\}$ such that E induces an associative product μ_E on the bar construction BA converting it into a dg Hopf algebra, and, similarly, E' induces a product $\mu_{E'}$ (not necessarily associative) on the double bar construction B^2A. (A major component of E' is a binary product $E'_{1,1}$ on A measuring the non-commutativity of the operation $E_{1,1}$; note that in the topological setting of A such a binary product is just provided by Steenrod's cochain $\smile_2$-product.) Below the algebra A is referred to as a *special* Hirsch algebra. Let $A_{\Bbbk}=A\otimes_{\mathbb{Z}}\Bbbk$.

We have the following theorem whose proof appears in Section 5:

Theorem 2. *Assume that $H^*(A_{\Bbbk})$ is finitely generated as a $\Bbbk$-module and that the Hochschild homology $HH_*(A_{\Bbbk})$ has finite type. Let $\varsigma_i(A_{\Bbbk})$ denote the cardinality of a minimal generating set of $HH_i(A_{\Bbbk})$. Then the integers $\varsigma_i(A_{\Bbbk})$ grow unbounded if and only if $\tilde{H}^*(A_{\Bbbk})$ requires at least two algebra generators.*

Let $C^*(X;\Bbbk)=C^*(\mathrm{Sing}^1X;\Bbbk)/C^{>0}(\mathrm{Sing}\ x\,;\Bbbk)$ in which $\mathrm{Sing}^1X\subset \mathrm{Sing}X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X. Theorem 1 is deduced from Theorem 2 by setting $A_{\Bbbk}=C^*(X;\Bbbk)$: Indeed, $C^*(X;\Bbbk)$ is a special Hirsch algebra ([2], [9]) and there is the isomorphism $HH_*(C^*(X;\Bbbk))\approx H^*(\Lambda X;\Bbbk)$ ([7], [18]). When $\tilde{H}^*(A_{\Bbbk})$ requires at least two algebra generators, we construct two infinite sequences in the Hochschild homology $HH_*(A_{\Bbbk})$ and take all possible products of their components to detect a submodule of $HH_*(A_{\Bbbk})$ at least as large as the polynomial algebra $\Bbbk[x,y]$. The construction of these sequences is in fact based on the notion of a formal ∞-implication sequence [17] that generalizes W. Browder's notion of ∞-implications [3]. As in [17], we use a filtered Hirsch model of $A_{\Bbbk}$ this time to construct a small model for the Hochschild chain complex of $A_{\Bbbk}$ and then to reduce the chain product [18] on this model inducing the aforementioned product on $HH_*(A_{\Bbbk})$. While the constructions of the sequences in our both papers are similar, here is an essential exception that we have to detect a desired sequence in the kernel of the canonical homomorphism $HH_*(A_{\Bbbk})\to H^*(BA_{\Bbbk})$ (corresponding to $i^*:H^*(\Lambda X;\Bbbk)\to H^*(\Omega X;\Bbbk)$ for $i:\Omega X\hookrightarrow\Lambda X$); also some technical details are simplified.

Though the author has been considered several special cases of Theorem 1 during the last two decades but it was just recently the integer coefficients come into play: In particular, the filtered Hirsch model over the integers controls the subtleties when dealing with the Bockstein homomorphism in question.

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2. Some preliminaries and conventions

We adopt some basic notations and terminology of [16]. We fix a ground commutative ring $\Bbbk$ with unit and let $\mu>0$ denote the smallest integer such that $\mu\kappa=0$ for all $\kappa\in\Bbbk$. When such a positive integer does not exist, we assume $\mu=0$. Let $A^*=\tilde{A}\oplus\Bbbk$ be a supplemented dga. In general A^* may be graded over the integers $\mathbb{Z}$. Assuming A to be associative, the (reduced) bar construction BA is

the tensor coalgebra $T(\bar{A})$, $\bar{A}=s^{-1}\tilde{A}$, with differential $d_{BA}=d_1+d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n]\in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\leq i\leq n}(-1)^{\epsilon^a_{i-1}}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n]$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\leq i<n}(-1)^{\epsilon^a_i}[\bar{a}_1|\cdots|\overline{a_ia_{i+1}}|\cdots|\bar{a}_n],$$

where $\epsilon^x_i=|x_1|+\cdots+|x_i|+i$. An associative dga A equipped with multilinear maps $E=\{E_{p,q}\}_{p+q>0}$,

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\qquad E_{1,0}=Id=E_{0,1},\ E_{p>1,0}=0=E_{0,q>1},$$

of degree $1-p-q$ is a *Hirsch* algebra if E lifts to a dg coalgebra map $\mu_E: BA\otimes BA\to BA$. A basic ingredient of E is the binary operation $\smile_1:=E_{1,1}$ measuring the non-commutativity of the $\cdot$ product on A by the formula

$$d(a\smile_1 b)-da\smile_1 b+(-1)^{|a|}a\smile_1 db=(-1)^{|a|}ab-(-1)^{|a|(|b|+1)}ba.$$

Given a Hirsch algebra $(A,\{E_{p,q}\})$ with $H=H(A)$, there is its filtered Hirsch model

$$f:(RH,d_h)\to(A,d_A) \tag{2.1}$$

in which $\rho:(RH,d)\to H$ is a multiplicative resolution of the commutative graded algebra (cga) H : As a module each row of R^*H^* for $m\in\mathbb{Z}$

$$\cdots\xrightarrow{d}R^{-2}H^m\xrightarrow{d}R^{-1}H^m\xrightarrow{d}R^0H^m\xrightarrow{\rho}H^m$$

represents a free resolution of a $\Bbbk$-module H^m. As an algebra $R^*H^*=T(V^{*,*})$ is a (bi)graded tensor algebra with

$$V^{*,*}=\mathcal{E}^{*,*}\oplus U^{*,*}=\mathcal{E}^{*,*}\oplus\mathcal{T}^{*,*}\oplus\mathcal{M}^{*,*};$$

the module $V^{0,*}=\mathcal{M}^{0,*}$ corresponds to a choice of multiplicative generators of H, while $\mathcal{M}^{-1,*}$ to relations among them which is not a consequence of that of the commutativity of the algebra H, and then $\mathcal{M}^{-r,*}$ for $r\geq 2$ is defined by the syzygies. The module $\mathcal{E}^{<0,*}$ just corresponds to the commutativity relation in H and is formed by the products under all operations $E_{p,q}$ on RH. In particular,

$$V^{-1,*}=\mathcal{E}^{-1,*}\oplus\mathcal{M}^{-1,*}$$

where $\mathcal{E}^{-1,*}$ is formed by the products $a\smile_1 b$ for $a,b\in R^0H^*$, while $\mathcal{M}^{-1,*}\neq 0$ for H to be a non-free cga (e.g. $\dim H^*<\infty$ and $H^{ev}\neq 0$; see (4.13)–(4.15) below). The module $\mathcal{T}$ is determined by the $\cup_2$-product that measures the non-commutativity of the $\smile_1$-product, so that its first non-trivial component $\mathcal{T}^{-2,*}$ contains the products $a\cup_2 b$ for $a,b\in V^{0,*}$. More precisely, (RH,d) is also endowed with Steenrod's type binary operation, denoted by $\smile_2$, so that the (minimal) Hirsch resolution (RH,d) can be viewed as a special Hirsch algebra with $E=\{E_{p,q}\}$ and E' consisting of a single operation $\smile_2:=E'_{1,1}$. In particular, the relationship between $a\cup_2 b$ and $a\smile_2 b$ for $a,b\in\mathcal{V}$ with $da,db\in V$, where $\mathcal{V}$ is a basis of V, is given by

$$a\smile_2 b=\begin{cases}a\cup_2 b, & a\neq b,\\ 2a\cup_2 a, & a=b;\end{cases}$$

thus $d(a \cup_2 a) = a \smile_1 a$ for a to be of even degree. ($a \cup_2 a = 0$ for an odd dimensional $a \in RH$.) Regarding the differential d_h on RH, we have

$$d_h = d + h, \quad h = h^2 + \cdots + h^r + \cdots, \quad h^r : R^pH^q \to R^{p+r}H^{q-r+1}.$$

Given $r \geq 2$, the map $h^r|_{R^{-r}H} : R^{-r}H \to R^0H$ is referred to as the *transgressive* component of h and is denoted by h^{tr}. The perturbation h is extended as a derivation on $\mathcal{E}$ so that $h^{tr}(\mathcal{E}) = 0$.

Furthermore, if A is also a special Hirsch algebra in (2.1), we can simply choose h and f such that

$$h^{tr}(a \cup_2 b) = 0 \ \text{ for } \ a \neq b \ \text{ in } \ \mathcal{V}. \tag{2.2}$$

Just to achieve this equality in (RH, d_h), we have in fact evoked the product $\mu_{E'}$ on B^2A (cf. [16, Proposition 4]).

A Hirsch resolution (RH, d) is *minimal* if

$$d(u) \in \mathcal{E} + \mathcal{D} + \kappa_u{\cdot}V \ \text{ for } \ u \in U$$

where $\mathcal{D}^{*,*} \subset R^*H^*$ denotes the submodule of decomposables $RH^+ \cdot RH^+$ and $\kappa_u \in \Bbbk$ is non-invertible; for example, $\kappa_u \in \mathbb{Z} \setminus \{-1, 1\}$ when $\Bbbk = \mathbb{Z}$ and $\kappa_u = 0$ for all u when $\Bbbk$ is a field.

In the sequel A denotes a torsion free special Hirsch $\mathbb{Z}$-algebra, while $A_\Bbbk = A \otimes_\mathbb{Z} \Bbbk$ and $H_\Bbbk = H(A_\Bbbk)$. Assume (RH, d) is minimal and let $RH_\Bbbk = RH \otimes_\mathbb{Z} \Bbbk$; in particular, $RH_\Bbbk = T(V_\Bbbk)$ for $V_\Bbbk = V \otimes_\mathbb{Z} \Bbbk$. When $\Bbbk$ is a field of characteristic zero, $H_\Bbbk = H \otimes \Bbbk$ and $\rho_\Bbbk = \rho \otimes 1 : RH_\Bbbk \to H_\Bbbk$ is a Hirsch resolution of $H_\Bbbk$, which is *not* minimal when $\operatorname{Tor} H \neq 0$. Assuming A is $\mathbb{Z}$-algebra in (2.1) we obtain a Hirsch model of $(A_\Bbbk, d_{A_\Bbbk})$ as

$$f_\Bbbk = f \otimes 1 : (RH_\Bbbk, d_h \otimes 1) \to (A_\Bbbk, d_{A_\Bbbk}).$$

2.1. **Small Hirsch resolution.** In practice it is convenient to reduce the Hirsch resolution RH at the cost of the module $\mathcal{E}$. Here we define such a small resolution $R_\tau H$ (compare with $R_\varsigma H$ in [16]). Namely, let

$$R_\tau H = RH/J_\tau$$

where $J_\tau \subset RH$ is a Hirsch ideal generated by

$$\{E_{p,q}(a_1, ..., a_p; a_{p+1}, ..., a_{p+q}),\, dE_{1,2}(a_1; a_2, a_3),\, dE_{2,1}(a_1, a_2; a_3),\, a \cup_2 b,\, d(a \cup_2 b) \\ \mid (p,q) \neq (1,1),\, a, b \in \mathcal{V},\, a \neq b\}$$

where $a_i \in RH$ unless $i = p + q$ for $p \geq 2$ and $q = 1$ in which case $a_{p+1} \in \mathcal{V}$. Since $d : J_\tau \to J_\tau$, we get a Hirsch algebra surjection $g_\tau : (RH, d) \to (R_\tau H, d)$ so that a resolution map $\rho : RH \to H$ factors as

$$\rho : (RH, d) \xrightarrow{g_\tau} (R_\tau H, d) \xrightarrow{\rho_\tau} H.$$

By definition we have $h : \mathcal{E} \to \mathcal{E}$; this fact together with (2.2) implies $h : J_\tau \to J_\tau$. Thus g_τ extends to a quasi-isomorphism of Hirsch algebras

$$g_\tau : (RH, d_h) \to (R_\tau H, d_h).$$

We have that the Hirsch algebra structure of $(R_\tau H, d_h)$ is given by the $\smile_1$-product satisfying the following two formulas. The (left) Hirsch formula: For $a, b, c \in R_\tau H$,

$$c \smile_1 ab = (c \smile_1 a)b + (-1)^{(|c|+1)|a|} a(c \smile_1 b) \tag{2.3}$$

and the (right) generalized Hirsch formula: For $a,b \in R_\tau H$ and $c \in V_\tau$ with $d_h(c) = \sum c_1 \cdots c_q$, $c_i \in V_\tau$,

$$(2.4)\quad ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (-1)^{\varepsilon_1}(a \smile_1 c)b, & q = 1, \\ \\ a(b \smile_1 c) + (-1)^{\varepsilon_1}(a \smile_1 c)b \\ \qquad + \sum\limits_{1 \le i < j \le q} (-1)^{\varepsilon_2}\, c_1 \cdots c_{i-1}(a \smile_1 c_i)c_{i+1} \\ \qquad\qquad \cdots c_{j-1}(b \smile_1 c_j)c_{j+1} \cdots c_q, & q \ge 2, \end{cases}$$

$$\varepsilon_1 = |b|(|c|+1),\ \varepsilon_2 = (|a|+1)(\epsilon^c_{i-1} + i + 1) + (|b|+1)(\epsilon^c_{j-1} + j + 1).$$

Remark 1. *1. Formula (2.4) can be thought of as a generalization of Adams' formula for the* $\smile_1$*-product in the cobar construction* [1, p. 36] *from* $q = 2$ *to any* $q \ge 2$.

2. We just pass from RH *to* $R_\tau H$ *to have formulas (2.3)–(2.4) therein; more precisely, we use them together with the commutativity of* $a \smile_1 b$ *for* $a, b \in V_\tau$ *to build the sequence given by (4.5) below.*

2.2. **Cohomology operation $\mathcal{P}_1$.** Let $\mu \ge 2$. Given an element $a \in A_\Bbbk$ and the integer $n \ge 2$, take (the right most) n^{th}-power of $\bar{a} \in \bar{A}_\Bbbk \subset BA_\Bbbk$ under the μ_E product on $BA_\Bbbk$ and then consider its component in $\bar{A}_\Bbbk$. Denote this component by $s^{-1}(a^{\uplus n})$ for $a^{\uplus n} \in A_\Bbbk$. The element $a^{\uplus n}$ has the form

$$a^{\uplus n} = a^{\smile_1 n} + Q_n(a),$$

where $Q_n(a)$ is expressed in terms of $E_{1,k}$ for $1 < k < n$ so that $Q_2(a) = 0$, i.e., $a^{\uplus 2} = a^{\smile_1 2}$. In particular, if $E_{1,k} = 0$ for $k \ge 2$ (e.g. $A_\Bbbk$ is a homotopy Gerstenhaber algebra (HGA)), then $a^{\uplus n} = a^{\smile_1 n}$.

Let $p \ge 2$ be the smallest prime that divides μ. Define the cohomology operation $\mathcal{P}_1$ on $H_\Bbbk$ as follows.

For p odd:

$$\mathcal{P}_1 : H^{2m+1}_\Bbbk \to H^{2mp+1}_\Bbbk, \quad [a] \to \left[\frac{\mu}{p} a^{\uplus p}\right],$$

for $p = 2$:

$$\mathcal{P}_1 : H^m_\Bbbk \to H^{2m-1}_\Bbbk, \qquad [a] \to \left[\frac{\mu}{2} a \smile_1 a\right], \quad da = 0,\ a \in A_\Bbbk.$$

Since $a \smile_1 a$ is a cocycle in $A_\Bbbk$ for an even dimensional cocycle a independently on the parity of p, we also set $\mathcal{P}_1[a] = [a \smile_1 a]$ for p odd and $\mu \ge 0$; obviously, $\mathcal{P}_1[a] = 0$. Let $\mathcal{P}^{(m)}_1$ denote the m-fold composition $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_1$. Given $x \in H_\Bbbk$, let $\nu \ge 0$ be the smallest integer such that $\mathcal{P}^{(\nu+1)}_1(x) = 0$. The integer ν is referred to as $\smile_1$*-height* of x.

3. Small model for the Hochschild chain complex

Given an associative dga C, its (normalized) *Hochschild chain complex* ΛC is $C \otimes BC$ with differential $d_{\Lambda C}$ defined by $d_{\Lambda C} = d_C \otimes 1 + 1 \otimes d_{BC} + \theta^1 + \theta^2$, where

$$\theta^1(u \otimes [\bar{a}_1|\cdots|\bar{a}_n]) = -(-1)^{|u|} u a_1 \otimes [\bar{a}_2|\cdots|\bar{a}_n],$$
$$\theta^2(u \otimes [\bar{a}_1|\cdots|\bar{a}_n]) = (-1)^{(|a_n|+1)(|u|+\epsilon^a_{n-1})} a_n u \otimes [\bar{a}_1|\cdots|\bar{a}_{n-1}].$$

The homology of ΛC is called the Hochschild homology of C and is denoted by $HH_*(C)$.

Let $C=T(V_{\Bbbk})$ be a tensor algebra with $V^*_{\Bbbk}$ a free $\Bbbk$-module. Denote

$$\bar V_{\Bbbk}=s^{-1}(V^{>0}_{\Bbbk})\oplus\Bbbk.$$

Then ΛC can be replaced by the small complex $(C\otimes \bar V_{\Bbbk},d_\omega)$ where the differential d_ω is defined as follows (cf. [21], [8]):

$$\begin{aligned}d_\omega(u\otimes\bar a)=d_{_C}(u)\otimes\bar a-(-1)^{|u|}(1\otimes s^{-1})\chi(u\otimes d_{_C}(a))\\ -(-1)^{|u|+|a|}(ua-(-1)^{|a||u|}au)\otimes 1,\end{aligned}$$

in which

$$\chi: C\otimes C\to C\otimes V_{\Bbbk}$$

is a map given for $u\otimes a\in C\otimes C$ with $a=a_1\cdots a_n$, $a_i\in V_{\Bbbk}$, by

$$\chi(u\otimes a)=\begin{cases}0, & a=1,\\ u\otimes a, & n=1,\\ \sum\limits_{1\le i\le n}(-1)^{\varepsilon}a_{i+1}\cdots a_n\,u\,a_1\cdots a_{i-1}\otimes a_i, & n\ge 2,\end{cases}$$

$$\varepsilon=(|a_{i+1}|+\cdots+|a_n|)(|u|+|a_1|+\cdots+|a_i|).$$

There is a chain map

$$\phi:(\Lambda C,d_{\Lambda C})\to(C\otimes\bar V_{\Bbbk},d_\omega) \tag{3.1}$$

defined for $u\otimes x\in\Lambda C$ by

$$\phi(u\otimes x)=\begin{cases}u\otimes 1, & x=[\,],\\ (-1)^{|u|}(1\otimes s^{-1})\chi(u\otimes a), & x=[\bar a],\\ 0, & x=[\bar a_1|\cdots|\bar a_n],\quad n\ge 2,\end{cases}$$

and ϕ is a homology isomorphism.

For $C=RH_{\Bbbk}$, define the differential $\bar d_h$ on $\bar V_{\Bbbk}$ by the restriction of d_h to $V_{\Bbbk}$ to obtain the cochain complex $(\bar V_{\Bbbk},\bar d_h)$. There are the sequences of maps

$$(\bar V_\tau)_{\Bbbk}\xleftarrow{\psi}B(R_\tau H_{\Bbbk})\xleftarrow{B(g_\tau)_{\Bbbk}}B(RH_{\Bbbk})\xrightarrow{Bf_{\Bbbk}}BA_{\Bbbk}$$

and

$$R_\tau H\otimes(\bar V_\tau)_{\Bbbk}\xleftarrow{\phi}\Lambda(R_\tau H_{\Bbbk})\xleftarrow{\Lambda(g_\tau)_{\Bbbk}}\Lambda(RH_{\Bbbk})\xrightarrow{\Lambda f_{\Bbbk}}\Lambda A_{\Bbbk}$$

subjected to the following proposition

Proposition 1. *There are isomorphisms of $\Bbbk$-modules*

$$\begin{aligned}H^*((\bar V_\tau)_{\Bbbk},\bar d_h)\underset{\approx}{\xleftarrow{\psi^*}}H^*(B(R_\tau H_{\Bbbk}),d_{_{B(R_\tau H_{\Bbbk})}})\underset{\approx}{\xleftarrow{B(g_\tau)^*_{\Bbbk}}}H^*(B(RH_{\Bbbk}),d_{_{B(RH_{\Bbbk})}})\\ \underset{\approx}{\xrightarrow{Bf^*_{\Bbbk}}}H^*(BA_{\Bbbk},d_{_{BA_{\Bbbk}}})\end{aligned}$$

and

$$\begin{aligned}H^*(R_\tau H\otimes(\bar V_\tau)_{\Bbbk},d_\omega)\underset{\approx}{\xleftarrow{\phi^*}}H^*(\Lambda(R_\tau H_{\Bbbk}),d_{_{\Lambda(R_\tau H_{\Bbbk})}})\underset{\approx}{\xleftarrow{\Lambda(g_\tau)^*_{\Bbbk}}}H^*(\Lambda(RH_{\Bbbk}),d_{_{\Lambda(RH_{\Bbbk})}})\\ \underset{\approx}{\xrightarrow{\Lambda f^*_{\Bbbk}}}H^*(\Lambda A,d_{_{\Lambda A}}).\end{aligned}$$

Note that the isomorphism ψ^* above is a consequence of a general fact about tensor algebras [4]. Recall also the following isomorphisms $H^*(BC^*(X;\Bbbk),d_{_{BC}})\approx H^*(\Omega X;\Bbbk)$ ([2]) and $H^*(\Lambda C^*(X;\Bbbk),d_{_{\Lambda C}})\approx H^*(\Lambda X;\Bbbk)$ ([7], [18]) to deduce the following proposition for $A_{\Bbbk}=C^*(X;\Bbbk)$.

Proposition 2. *There are isomorphisms of $\Bbbk$-modules*

$$H^*((\bar{V}_\tau)_{\Bbbk}, \bar{d}_h) \approx H^*(BC^*(X;\Bbbk), d_{BC}) \approx H^*(\Omega X;\Bbbk)$$

and

$$H^*(R_\tau H \otimes (\bar{V}_\tau)_{\Bbbk}, d_\omega) \approx H^*(\Lambda C^*(X;\Bbbk), d_{\Lambda C}) \approx H^*(\Lambda X;\Bbbk).$$

In the sequel by abusing the notations we will denote $R_\tau H$ again by RH.

3.1. **Product on the small model of the Hochschild chain complex.** Let $RH = T(V)$ be the (small) Hirsch resolution with only $\smile_1$-product. First, define a product on $\bar{V}$ for $\bar{a}, \bar{b} \in \bar{V}$ by

$$\bar{a}\bar{b} = \overline{a \smile_1 b} \quad \text{with} \quad \bar{a}1 = 1\bar{a} = \bar{a}.$$

Next define a product on $RH \otimes \bar{V}$ for $u \otimes \bar{a}, v \otimes \bar{b} \in RH \otimes \bar{V}$ with $d_h a = \sum a_1 a_2$ mod V, $d_h b = \sum b_1 b_2$ mod V and $a_2, b_1 \in V$, $a_1, b_2 \in RH$ by

$$\begin{aligned}(u \otimes \bar{a})(v \otimes \bar{b}) = (-1)^{\epsilon_1} uv \otimes \overline{a \smile_1 b} + u(a \smile_1 v) \otimes \bar{b} + (-1)^{\epsilon_2} (u \smile_1 b)v \otimes \bar{a} \\ + (-1)^{\epsilon_3} (u \smile_1 b)(a \smile_1 v) \otimes 1 \\ - \sum (-1)^{\epsilon_4} u(a_1 \smile_1 v) \otimes \overline{a_2 \smile_1 b} + (-1)^{\epsilon_5} (u \smile_1 b)(a_1 \smile_1 v) \otimes \bar{a}_2 \\ + \sum (-1)^{\epsilon_6} (u \smile_1 b_2) v \otimes \overline{a \smile_1 b_1} + (-1)^{\epsilon_7} (u \smile_1 b_2)(a \smile_1 v) \otimes \bar{b}_1,\end{aligned}$$

$$\begin{aligned} &\epsilon_1 = (|a|+1)|v|, && \epsilon_4 = |a_1||b| + (|a_2|+1)|v|, \\ &\epsilon_2 = |a|(|v|+|b|+1) + |v||b|, && \epsilon_5 = |a_2|(|v|+1) + (|a|+|v|)|b|, \\ &\epsilon_3 = (|a|+|v|)(|b|+1), && \epsilon_6 = (|a|+|b_2|)(|v|+1) + (|a|+|b_1|)|b_2|, \\ & && \epsilon_7 = (|a|+|v|)(|b_2|+1) + (|b_1|+1)|b_2|, \end{aligned}$$

and

$$\begin{aligned} (u \otimes 1)(v \otimes 1) &= uv \otimes 1 \\ (u \otimes 1)(v \otimes \bar{b}) &= uv \otimes \bar{b} + (-1)^{(|v|+1)(|b|+1)} (u \smile_1 b)v \otimes 1, \\ (u \otimes \bar{a})(v \otimes 1) &= (-1)^{(|a|+1)|v|} uv \otimes \bar{a} + u(a \smile_1 v) \otimes 1. \end{aligned}$$

Formulas (2.3)–(2.4) guarantee that such defined products satisfy the Leibniz rule, and we obtain a short sequence of dg algebras

$$\bar{V}_{\Bbbk} \overset{\pi}{\leftarrow} RH \otimes \bar{V}_{\Bbbk} \overset{\iota}{\leftarrow} RH_{\Bbbk} \tag{3.2}$$

with ι and π the standard inclusion and projection respectively.

Remark 2. *1. The dga $(RH \otimes \bar{V}_{\Bbbk}, d_\omega)$ can be thought of as a non-commutative version of the cdga $\left(\mathcal{A}^*(X) \otimes H^*(\Omega X;\mathbb{Q}), \bar{d}\right)$ modeling ΛX* [20].

2. Taking into account the product on the Hochschild chain complex ΛA of $A = C^*(X;\Bbbk)$ defined in [18] *one can show that the map ϕ given by (3.1) is multiplicative up to homotopy; thus the sequence given by (3.2) provides a small multiplicative model of the free loop fibration.*

4. Canonical sequences in $RH \otimes \bar{V}$

Motivated by the notion of a formal ∞-implication sequence [17] here we construct certain sequences in the dga $(RH \otimes \bar{V}, d_\omega)$ used in the proof of Theorem 2. First, we consider a more general situation.

4.1. **The sequence $\mathbf{x}_\mu$ in (C, d_C).** Let (C^*, d_C) be a cochain complex of torsion free abelian groups and let

$$t_{\Bbbk} : C \to C_{\Bbbk} \ (= C \otimes_{\mathbb{Z}} \Bbbk)$$

be the standard map. Let $x \in C$ be a mod μ cocycle, i.e., $d_{C_{\Bbbk}}(t_{\Bbbk}x) = 0$. Consider for x the following two conditions:

$$[x] \neq 0 \text{ in } H(C) \text{ for } d_C x = 0; \tag{4.1}$$

(4.2) If $[t_{\Bbbk}x] = 0 \in H(C_{\Bbbk})$, i.e., there is a relation $d_C a = x + \lambda a'$, $a, a' \in C$, then
$$d_{C_{\Bbbk}}(t_{\Bbbk}a') = 0 \ \text{ for } \lambda \text{ to be the greatest integer divisible by } \mu.$$

Obviously, for x with $d_C x = 0$ condition (4.2) follows from (4.1). In any case for a' from (4.2) we have that $[t_{\Bbbk}a'] \neq 0$ in $H(C_{\Bbbk})$.

Let $\mathbf{x} = \{x(n)\}_{n\geq 0}$ be a sequence in C^* with $|x(k)| \neq |x(\ell)|$ for $k \neq \ell$ and let $x(n)$ satisfy (4.1)–(4.2) for all n in which case we also say that $\mathbf{x}$ satisfies (4.1)–(4.2). Define the *associated* sequence $\mathbf{x}_\mu = \{x_\mu(n)\}_{n\geq 0}$ in C as follows: Given $n \geq 0$, let

$$x_\mu(n) = \begin{cases} a'_n, & [t_{\Bbbk}x(n)] = 0, \\ x(n), & [t_{\Bbbk}x(n)] \neq 0, \end{cases}$$

where a'_n is resolved from (4.2) for $x = x(n)$. Obviously, $[t_{\Bbbk}x_\mu(n)] \neq 0$ in $H(C_{\Bbbk})$ for all n.

A pair of sequences $(\mathbf{x}, \mathbf{y}') = (\{x(i)\}_{i\geq 0}, \{y'(i)\}_{i\geq 0})$ satisfying (4.1)–(4.2) is said to be *admissible*, if $\alpha_1 x(i) + \alpha_2 y'(j)$ also satisfies (4.1)–(4.2) whenever $|x(i)| = |y'(j)|$, $\alpha_1, \alpha_2 \in \mathbb{Z}$. Then obtain the sequence $\mathbf{y}_\mu = \{y_\mu(j)\}_{j\geq 0}$ from an admissible pair $(\mathbf{x}, \mathbf{y}')$ as follows. Given $j \geq 0$, set

$$y_\mu(j) = \begin{cases} a'_j, & [t_{\Bbbk}(\alpha_1 x(i) + \alpha_2 y'(j))] = 0, \\ y'_\mu(j), & \text{otherwise}, \end{cases} \tag{4.3}$$

where a'_j is resolved from (4.2) for $x = \alpha_1 x(i) + \alpha_2 y'(j)$; in particular, the pair $([t_{\Bbbk}x_\mu(i)], [t_{\Bbbk}y_\mu(j)])$ is linearly independent in $H(C_{\Bbbk})$.

4.2. **Sequences in $(RH \otimes \bar{V}, d_\omega)$.** Let $\mathcal{D}_{\Bbbk} \subset RH$ be a subset defined by

$$\mathcal{D}_{\Bbbk} = \{u + \mu v \mid u \in \mathcal{D}, v \in V\}.$$

An element $x \in V$ with $d_h x \in \mathcal{D} + \lambda V$, $\lambda \neq 1$, is λ-*homologous to zero* if there are $u, v \in V$ and $c \in \mathcal{D}$ such that

$$d_h u = x + c + \lambda v.$$

x is *weakly homologous* to zero if $v = 0$ above. We have the following statement (cf. [17]):

Proposition 3. *Let $v \in V$ and $d_h v \in \mathcal{D}$. If $d_h v$ has a summand component $v_1 v_2 \in \mathcal{D}$ such that $v_1, v_2 \in V$, $d_h v_1, d_h v_2 \in \mathcal{D}$, both v_1 and v_2 are not weakly homologous to zero, then v is also not weakly homologous to zero.*

Note also that under the hypotheses of the proposition if $[\bar{v}_1], [\bar{v}_2] \neq 0$, then $[\bar{v}] \neq 0$ in $H^*(\bar{V}, \bar{d}_h)$; for example, for $RH = \Omega BH_{\Bbbk}$ (the cobar-bar construction of $H_{\Bbbk}$) with $V = BH_{\Bbbk}$ and $\Bbbk$ a field, the proposition reflects the obvious fact that an element $x \in H^*(BH_{\Bbbk})$ is non-zero whenever some $x' \otimes x'' \neq 0$ in $\Delta x = \sum x' \otimes x''$ for the coproduct $\Delta : BH_{\Bbbk} \to BH_{\Bbbk} \otimes BH_{\Bbbk}$.

Let

$$\chi_1 : RH \to RH \otimes \bar{V}$$

be a map defined for $a \in RH$ by $\chi_1(a) = \phi(1 \otimes [\bar{a}])$, where ϕ is given by (3.1), and define two subsets $\widetilde{\mathcal{D}}, \widetilde{\mathcal{D}}_{\Bbbk} \subset \mathcal{D}_{\Bbbk}$ as

$$\widetilde{\mathcal{D}} = \{a \in \mathcal{D}_{\Bbbk} \mid \chi_1(a) = 0\} \text{ and } \widetilde{\mathcal{D}}_{\Bbbk} = \{a \in \mathcal{D}_{\Bbbk} \mid \chi_1(a) = 0 \mod \mu\}$$

(e.g. $\widetilde{\mathcal{D}}$ contains the expressions of the form $ab - (-1)^{(|a|+1)(|b|+1)}ba$ and also of the form y^{λ} with $|y|$ odd and λ even, while $\widetilde{\mathcal{D}}_{\Bbbk}$ contains y^{λ} with $|y|$ even and λ divisible by $\mu \geq 2$). Given $a \in RH$, obviously $d_h a \in \widetilde{\mathcal{D}}$ implies $\chi_1(a) \in \operatorname{Ker} d_{\omega}$, while $d_h a \in \widetilde{\mathcal{D}}_{\Bbbk}$ implies $d_{\omega}\chi_1(a) = 0 \mod \mu$.

The following statement is also simple.

Proposition 4. *Given $a \in RH$, let $d_h a \in \widetilde{\mathcal{D}}$. If $a = v_1 v_2 + c$ such that c does not contain $\pm v_2 v_1$ as a summand component and $v_1 v_2$ does not occur as a summand component of $d_h w$ for any $w \in RH$ unless $w \in \mathcal{E}$ or $d_h w$ has a summand component from V, too, then $[\chi_1(a)] \neq 0$ in $H(RH \otimes \bar{V}, d_{\omega})$.*

For $v_1 = 1$, $v_2 \in V$ and $c = 0$, taking into account (3.2) the proposition in particular implies that if v_2 is not λ-homologous to zero, then $[\chi_1(v_2)] = [1 \otimes \bar{v}_2] \neq 0$ in $H(RH \otimes \bar{V}, d_{\omega})$.

By the universal coefficient theorem we have an isomorphism

$$H^n_{\Bbbk} \approx H^n \otimes \Bbbk \bigoplus Tor(H^{n+1}, \Bbbk) := H^n_{\Bbbk,0} \bigoplus H^n_{\Bbbk,1}.$$

Given $\mu \geq 2$, define a subset $K_{\mu} \subset V^{-1,*}$ as

$$K_{\mu} = \left\{a \in n \cdot \mathcal{V}^{-1,*} \,|\, da = \lambda b \neq 0,\, b \in R^0 H^*,\, \mu \text{ divides } \lambda,\, 1 \leq n < \mu\right\}$$

(i.e., $n = 1$ for μ to be a prime) and let

$$\Xi_{\mu} = K_{\mu} \cup \mathcal{V}^{0,*}.$$

Let $\mathcal{H}_{\Bbbk} \subset H_{\Bbbk}$ denote a (minimal) set of multiplicative generators. Given $x \in H_{\Bbbk}$, let x_0 be its *representative* in RH with $[t_{\Bbbk} x_0] = x$; in particular, $x_0 \in R^0 H^*$ for $x \in H_{\Bbbk,0}$, while $x_0 \in \mathcal{V}^{0,*}$ or $x_0 \in K_{\mu}$ when $x \in \mathcal{H}_{\Bbbk,0}$ or $x \in \mathcal{H}_{\Bbbk,1}$ respectively; given $x \in H_{\Bbbk,1}$ and $y \in H_{\Bbbk,0}$ with $dx_0 = \lambda y_0$ and λ divisible by μ, we also denote such a connection between x and y as

$$\beta_{\lambda}(x) = y,$$

where β_{μ} is the Bockstein cohomology homomorphism associated with the sequence $0 \to \mathbb{Z}_{\mu} \to \mathbb{Z}_{\mu^2} \to \mathbb{Z}_{\mu} \to 0$ and is simply denoted by β.

On the other hand, if

$$\sigma : H^*(A_{\Bbbk}) \to H^{*-1}(BA_{\Bbbk}), \quad [a] \to [\bar{a}]$$

is the cohomology suspension map, $x \in \operatorname{Ker} \sigma$ is equivalent to say that x_0 is λ-homologous to zero in (RH, d_h).

Let $O_{\Bbbk} \subset R^0 H$ be a subset given by

$$O_{\Bbbk} = \left\{b \in R^0 H \,|\, da = \theta b \text{ for } a \in V^{-1,*} \text{ and } \theta \in \mathbb{Z} \text{ is prime with } \mu\right\}.$$

Obviously $\rho_{\Bbbk}(O_{\Bbbk}) = 0$ and let $\mathcal{O}_{\Bbbk} \subset RH$ be a Hirsch ideal generated by $O_{\Bbbk}$. In the sequel we consider the quotient Hirsch algebra $RH/\mathcal{O}_{\Bbbk}$ which by abusing the notations we will again denote by RH.

Given $w \in RH$, let $d_h w$ admit a decomposition

$$(4.4)\quad d_h w = w_1 + w_2, \quad w_1 \in \widetilde{\mathcal{D}}_{\Bbbk},\ w_2 = P(z_1, ..., z_q) \in \mathcal{D}_{\Bbbk} \ \text{ for } \ z_i \in \Xi_\mu, 1 \le i \le q.$$

Given $n \ge 0$ and assuming w to be odd dimensional, define an indecomposable element $x'(n) \in RH$ as

$$x'(n) = \begin{cases} w^{\smile_1(n+1)}, & w_2 = 0, \\ \frac{\mu}{p} w^{\smile_1(n+1)} \smile_1 Z_w + \gamma_w, & w_2 \ne 0, \qquad\qquad \mu \ge 2, \\ w^{\smile_1(n+1)} \smile_1 z_1 \cdots \smile_1 z_q + \gamma_w, & w_2 \ne 0, \quad |z_j| \text{ is even}, \quad \mu = 0, \end{cases}$$

where $Z_w = z_1{}^{\smile_1 (p^{\nu_1+1}-1)} \smile_1 \cdots \smile_1 z_q{}^{\smile_1(p^{\nu_q+1}-1)}$ with the convention that the component $z_j{}^{\smile_1(p^{\nu_j+1}-1)}$ is eliminated whenever $[z_j] = \mathcal{P}_1([z_i])$ for some $1 \le i < j \le q$, while γ_w is defined so that $d_h x'(n) \in \widetilde{\mathcal{D}}_{\Bbbk}$; namely, the existence of γ_w uses Hirsch formulas (2.3)–(2.4) and the fact that $\frac{\mu}{p} z_j{}^{\smile_1 p^{\nu_j+1}}$ (and $z_j \smile_1 z_j$ for $|z_j|$ even and $\mu = 0$) is mod μ cohomologous to zero for all j.

Then w rises to the sequence $\mathbf{x} = \{x(n)\}_{n\ge 0}$ in $RH \otimes \bar{V}$ defined by

$$(4.5)\qquad x(n) = \chi_{1}(x'(n))\,.$$

Thus $d_\omega x(n) = 0$ mod μ for all n.

In the sequel we apply to (4.5) for the following specific cases of w. First, given $x \in \mathcal{H}_{\Bbbk}$ and its representative $x_0 \in \Xi_\mu \subset RH$, the element $w := x_0$ obviously satisfies (4.4) (with $w_2 = 0$); thus for $x \in \mathcal{H}^{od}_{\Bbbk}$, equality (4.5) is specified as

$$x(n) = 1 \otimes s^{-1}\left(x_0{}^{\smile_1(n+1)}\right).$$

Example 1. *Let $\mu = 2$ and $x \in \mathcal{H}^{od}_{\Bbbk}$. When $\mathcal{P}_1(x) = 0$, we have a relation in (RH, d_h)*

$$d_h v = x_0 \smile_1 x_0 + 2x_1 \quad \text{with} \quad dx_1 = x_0^2, \quad x_1 \in V^{-1,*},\ v \in V^{-2,*}.$$

Therefore, $x_\mu(1) = \begin{cases} 1 \otimes \overline{x_0 \smile_1 x_0}, & \mathcal{P}_1(x) \ne 0,\ |x| \textit{ is the smallest}, \\ 1 \otimes \bar{x}_1, & \mathcal{P}_1(x) = 0 \end{cases}$ *in* $RH \otimes \bar{V}$.

Furthermore, $x \in \mathcal{H}^{od}_{\Bbbk}$ rises to the sequence $\{x_n \in V\}_{n\ge 0}$ in (RH, d) :

For $x \in \mathcal{H}_{\Bbbk,0}$ ($x_0 \in \mathcal{V}^{0,*}$),

$$(4.6)\quad dx_n = \sum_{\substack{i+j=n-1 \\ i,j\ge 0}} \varepsilon_{i,j} x_i x_j, \qquad \varepsilon_{i,j} = \begin{cases} 2, & \rho x_0^2 \ne 0, \quad i, j \text{ are even}, \\ 1, & \text{otherwise}, \end{cases} \qquad n \ge 1,$$

with $x_1 = -x_0 \smile_1 x_0$ when $\rho x_0^2 \ne 0$.

For $x \in \mathcal{H}_{\Bbbk,1}$ with $dx_0 = \lambda x'_0$ ($x_0 \in K_\mu$),

$$(4.7)\qquad dx_n = \sum_{\substack{i+j=n-1 \\ i,j\ge 0}} x_i x_j + \lambda x'_n, \qquad dx'_n = -\frac{1}{\lambda}\, d\left(\sum_{\substack{i+j=n-1 \\ i,j\ge 0}} x_i x_j\right), \quad n \ge 1.$$

The element x_n is of odd degree in (4.6)–(4.7) for all n. The action of h on x_n in (RH, d_h) is given by the following formula. Let $\tilde{x}_i = y_i + h^{tr} x_i$ with $y_i = 0$ or

$dy_i = -\lambda h^{tr} x'_i$ for x_i in (4.6) or (4.7) respectively; for $0 \le i_1 \le \cdots \le i_r < n$, $r \ge 2$, denote also $\tilde{x}_{i_1,...,i_r} = (-1)^r \tilde{x}_{i_1} \cup_2 \cdots \cup_2 \tilde{x}_{i_r}$.

$$hx_n = \tilde{x}_n + \sum_{i_1+\cdots+i_r+r=n+1} \varepsilon_{i_1,...,i_r} \left(\tilde{x}_{i_1,...,i_{r-1}} \smile_1 x_{i_r} - \tilde{x}_{i_1,...,i_r}\right), \tag{4.8}$$

$$\varepsilon_{i_1,...,i_r} = \begin{cases} 2, & x \in H_{\Bbbk,0},\ \rho x_0^2 \neq 0,\ \text{some } (i_s,i_t)_{1\le s,t\le r} \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

Now given $x \in H^{od}_{\Bbbk}$ and the smallest odd prime p that divides μ, recall the definition of the symmetric Massey product $\langle x \rangle^p$ [10] for which we have the Kraines formula (see also [16])

$$\beta \mathcal{P}_1(x) = -\langle x \rangle^p. \tag{4.9}$$

We also have the equality

$$\rho_{\Bbbk} h^{tr}(x_{p-1}) = -\langle x \rangle^p. \tag{4.10}$$

When $\mathcal{P}_1(x) \in H_{\Bbbk,0}$, we have $\beta\mathcal{P}_1(x) = 0 = \langle x \rangle^p$. Hence we obtain $\rho_{\Bbbk} h^{tr}(x_{p-1}) = 0$, and, consequently, $h^{tr}x_{p-1} = 0$. Then $dx_{p-1} \in \widetilde{\mathcal{D}}_{\Bbbk}$ implies that the element

$$w = x_{p-1} \tag{4.11}$$

satisfies (4.4).

4.3. **Element $\varpi \in V$ associated with a relation in $H_{\Bbbk}$.** Given any two multiplicative generators $a, b \in H_{\Bbbk,1}$ with $da_0 = \lambda_a a'_0$ and $db_0 = \lambda_b b'_0$, $a'_0, b'_0 \in V^{0,*}$, $a_0, b_0 \in K_\mu$, we have a relation in (RH, d)

$$du = a_0 b_0 + \lambda u' \quad \text{with} \quad du' = -\frac{\lambda_a}{\lambda} a'_0 b_0 - (-1)^{|a|} \frac{\lambda_b}{\lambda} a_0 b'_0, \tag{4.12}$$
$$\lambda = \mathrm{g.c.d.}(\lambda_a, \lambda_b), \quad u \in V^{-3,*},\ u' \in V^{-2,*}.$$

Regarding the action of h in (RH, d_h), we have $hu = (h^2 + h^3)u$ and, in particular, the relation $ab = 0$ in $H_{\Bbbk}$ is equivalent to the equalities $h^2 u = 0$ and $h^3 u = 0$ mod μ in (RH, d_h). More generally, a relation of the form $ab + c_1 + c_2 = 0$ in $H_{\Bbbk}$, where $c_1 = P_1(a_1, ..., a_{q_1}) = \sum_r \lambda_r a_{1,r}^{n_{1,r}} \cdots a_{p_r,r}^{n_{q_r,r}}$ with a single $a_{i,r} \in H_{\Bbbk,1}$ with $n_{i,r} = 1$ and the other $a_{j,r} \in H_{\Bbbk,0}$ for each r, and $c_2 = P_2(b_1, ..., b_{q_2})$ with $b_j \in H_{\Bbbk,0}$ for all j, yields that $h^2 u \in \mathcal{D}_{\Bbbk}$ and $h^3 u \in \mathcal{D}_{\Bbbk}$. If either a or b is from $H_{\Bbbk,0}$, then to $ab = 0$ corresponds the equality given by a similar formula as (4.12) but this time $(u, u') \in (V^{-2,*}, V^{-1,*})$ with $du' = -(-1)^{|a|} a_0 b'_0$ or $du' = -a'_0 b_0$ respectively. Since $a_0 b_0$ is mod μ cohomologous to hu, we have that for a given $1 \le k < r$, any cocycle in $\bigoplus_{0 \le i \le r} R^{-i}H$ is mod μ cohomologous to a cocycle in $\bigoplus_{0 \le i \le k} R^{-i}H$.

In general, consider a (homogeneous) multiplicative relation in $H^*_{\Bbbk}$

$$P(y_1, ..., y_q) = 0, \quad y_i \in \mathcal{H}_{\Bbbk} \tag{4.13}$$

which is not a consequence of the commutativity of the algebra $H_{\Bbbk}$ (and also is not decomposable by any other relations). Obviously $P(b_1,..., b_q)$ for $b_i = (y_i)_0 \in \Xi_\mu$ is a mod μ cohomologous to zero cocycle in (RH, d_h). If $P(b_1,..., b_q) \in \bigoplus_{0 \le i \le r} R^{-i}H$ with $r \ge 3$, then, as above, there is a mod μ cohomologous to $P(b_1,..., b_q)$ cocycle $P'(z_1, ..., z_m)$ that lies in $\bigoplus_{0 \le i \le 2} R^{-i}H$; in particular $z_i \in \Xi_\mu$ for all i, but each

monomial of $P'(z_1,...,z_m)$ may contain at most two variables z_i from K_μ. So that we have one of the following equalities in (RH,d_h) :

$$(4.14)\quad d_hu=\begin{cases} P(b_1,...,b_q), & u\in V^{-1,*}, \qquad\qquad y_i\in\mathcal{H}_{\Bbbk,0},\ 1\leq i\leq q,\\ P'(z_1,...,z_m), & u\in V^{-2,*},\\ P'(z_1,...,z_m)+\lambda u', & (u,u')\in (V^{-r,*},V^{-r+1,*}), \qquad r=2,3.\end{cases}$$

Note that $d_hu\notin\widetilde{\mathcal{D}}_\Bbbk$ (since $du\notin\widetilde{\mathcal{D}}_\Bbbk$), unless each monomial of $P(y_1,...,y_q)$ is of the form $\alpha_y y^\lambda$, $\alpha_y\in\mathbb{Z}$ with λ divisible by $\mu\geq 2$ in which case $d_hu\in\widetilde{\mathcal{D}}_\Bbbk$. In any case $w:=u$ obviously satisfies (4.4). For example, given $y\in\mathcal{H}_\Bbbk$ with $\dim H_\Bbbk<\infty$, let $\hbar_y$ be the height of y with respect to the product on $H_\Bbbk$ so that we have the relation $P(y)=y^{\hbar_y+1}=0$ and then (4.14) becomes the form

$$(4.15)\qquad d_hu=\begin{cases} y_0^{\hbar_y+1}, & y\in\mathcal{H}_{\Bbbk,0},\\ y_0'y_0+\lambda u', & y\in\mathcal{H}_{\Bbbk,1},\end{cases}$$

where $y_0'\in V^{0,*}\oplus V^{-1,*}$ is mod λ cohomologous to $y_0^{\hbar_y}\in R^{-\hbar_y}H^*$ in (RH,d_h) (in particular, $y_0'=y_0$ for $\hbar_y=1$; cf. (4.7)).

Remark 3. *Note that when both $|y|$ and μ are odd, always $\hbar_y=1$ and we say that (4.15) is a consequence of the commutativity of $H_\Bbbk$.*

Now given (4.13) of the smallest degree and assuming $\tilde{H}_\mathbb{Q}$ is either trivial or has a single algebra generator, we define an odd dimensional $\varpi\in V$ in the following three cases. In the case $\tilde{H}_\mathbb{Q}\neq 0$, denote a single multiplicative generator of infinite order of H by $\mathfrak{z}$ and let $z=t_\Bbbk^*(\mathfrak{z})$; thus $z=\mathfrak{z}\otimes 1\in H_{\Bbbk,0}$. (Warning: z may not be a multiplicative generator of $H_\Bbbk$.)

(i) When P is even dimensional in (4.13), u is odd dimensional in (4.14) and we set $\varpi=u$.

(ii) When P is odd dimensional in (4.13), u is even dimensional in (4.14) and we have to consider the following subcases. Suppose that the following expression

$$(4.16)\qquad \beta_\lambda(P(y_1,...,y_q))=\sum_{1\leq i\leq q}(-1)^{|y_1|+\cdots+|y_{i-1}|}P(y_1,...,\beta_\lambda(y_i),...,y_q)$$

is formally (i.e., independently on $H_\Bbbk$) trivial.

(ii_1) Let $y_i\in\mathcal{H}_{\Bbbk,0}$ for all i, i.e., $\beta_\lambda(y_i)=0$ and the corresponding relation of (4.13) in (RH,d_h) is given by the first equality of (4.14). Since either at most one y_i may be equal to z for $q\geq 2$ or $q=1$ and $y_1=z$ with $z\in\mathcal{H}^{od}_{\Bbbk,0}$ for μ even, it is easy to see that there is $c\in H^+_\Bbbk\cdot H^+_\Bbbk$ such that $\beta_\lambda(c)$, as a formal expression, is equal to $P(y_1,...,y_q)$. (In the last case $c=\lambda_z\mathcal{P}_1(z)z^{2m-1}$ for $P(z)=\lambda_z z^{2m+1}, m\geq 1$.) This situation answers to the following relations in (RH,d_h). There is a pair (b_c,w_c) with $b_c\in\mathcal{D}$, $w_c\in V$ such that

$$dw_c=-b_c+\lambda u\ \text{ with }\ db_c=\lambda P(b_1,...,b_q)$$

where $[t_\Bbbk h^{tr}w_c]=c$ and u is given by the first equality of (4.14). (In particular $\lambda=\mu$ and $b_c=-\lambda_z\frac{\mu}{2}(z_0\smile_1 z_0)z_0^{2m-1}$ when μ is even and $P(z)=\lambda_z z^{2m+1}$ as above.) Therefore, if $c=\mathfrak{c}\otimes 1$ with $\mathfrak{c}\in H$, then $\mathfrak{c}=\rho h^{tr}w_c$. Note also that $\mathfrak{c}$ must be indecomposable in H since there is no relation in degrees $<|P|$ in $H_\Bbbk$. When $\mathfrak{c}$ is of finite order, there is $a\in H_{\Bbbk,1}$ with $\beta_\lambda(a)=c$ so that $da_0=\lambda h^{tr}w_c$. Furthermore, when itself a is linearly dependent on $\mathcal{P}_1(x)$ for some $x\in H^{od}_\Bbbk$, i.e., $ka=\ell\,\mathcal{P}_1(x)$, $k,\ell\in\mathbb{Z}$, we obtain $\frac{k\lambda}{\mu}\,c=-\ell\,\langle x\rangle^p$ since (4.9). Taking into account (4.10) we have

that $\frac{k\lambda}{\mu}\, h^{tr} w_c$ is mod μ cohomologous to $\ell\, h^{tr} x_{p-1}$, i.e., there is $v \in V^{-1,*}$ with $dv = \frac{k\lambda}{\mu}\, h^{tr} w_c - \ell\, h^{tr} x_{p-1} + \mu v'$, $v' \in V^{0,*}$. Define $\varpi \in V$ by

$$\varpi = \begin{cases} \frac{k\lambda}{\mu}\, w_c - v - \ell\, x_{p-1}, & ka = \ell\, \mathcal{P}_1(x) \neq 0, \\ a_0, & \text{otherwise.} \end{cases} \tag{4.17}$$

Obviously $w := \varpi$ satisfies (4.4). Let now $\mathfrak{c}$ be of infinite order. Since $\mathfrak{z}$ is unique (when it exists), neither y_i occurs as z, so there is the other $\bar{c} \in H_{\Bbbk}$ such that $\beta_{\bar{\lambda}}(\bar{c})$, as a formal expression, is again equal to $P(y_1, ..., y_q)$. This time $\bar{\mathfrak{c}}$ is of finite order and hence the definition of w by means of formula (4.17) is not obstructed.

(ii_2) Let at least two y_i lie in $\mathcal{H}_{\Bbbk,1}$. Then the corresponding relation of (4.13) in (RH, d_h) is given by the second equality of (4.14). Since for each $y_i \in \mathcal{H}_{\Bbbk,1}$ with $\beta_\lambda(y_i) = y_i'$ either $y_i' \in \mathcal{H}_{\Bbbk,0}$ or $y_i' = z^n$ for $z \in \mathcal{H}_{\Bbbk,0}$, $n \geq 2$, there exist $c \in H_{\Bbbk}^{+} \cdot H_{\Bbbk}^{+}$ such that $\beta_\lambda(c)$, as a formal expression, is equal to $P(y_1, ..., y_q)$. When $\mathfrak{c}$ is of finite order, we can define w entirely analogously as in item (i), i.e., by formula (4.17); otherwise, for $\mathfrak{c}$ to be of infinite order, we get an obstruction, i.e., when we have

$$\beta_\lambda(c) = P(y_1, ..., y_q) = 0 \quad \text{for} \quad c = z \quad \text{modulo decomposables.} \tag{4.18}$$

Note also that $du \notin \widetilde{\mathcal{D}}_{\Bbbk}$ for u corresponding to this relation by (4.14).

(iii) Suppose that expression (4.16) is not formally trivial. Then the corresponding relation in (RH, d_h) is given by the third equality of (4.14). Consider two subcases.

(iii_1) Let (4.13) be specified as

$$P(y_1, .., y_q) = \lambda' b^n c\, x = 0 \ \text{with} \ \beta_\lambda(b) = c \ \text{for} \quad b \in \mathcal{H}_{\Bbbk,1}^{ev},\ c \in \mathcal{H}_{\Bbbk,0}^{od},\ x \in H_{\Bbbk,0},\ \lambda' \in \mathbb{Z},\ n \geq 1. \tag{4.19}$$

In particular $\beta_\lambda(\lambda' b^k x) = 0$ for $k > n$. Since $|b^{n+1}| < |b^n c|$ and (4.19) is chosen to be of the smallest degree, we have $\lambda' b^{n+1} x \neq 0$. Consider two elements a_i for $i = n+1, n+2$ with $\beta_\lambda(a_i) = 0$ where

$$a_i = \begin{cases} b^i, & i \text{ is divisible by } \mu \\ \lambda' b^i x, & \text{otherwise.} \end{cases}$$

When $a_{n+2} \neq 0$, there is $a \in \mathcal{H}_{\Bbbk,1}^{od}$ with $\beta_\lambda(a) = a_{n+1}$ or $\beta_\lambda(a) = a_{n+2}$ and we set $\varpi = a_0$. When $a_{n+2} = 0$, we consider this relation as a particular case of (4.13) and set $\varpi = u$ as in item (i) above.

(iii_2) When at least one monomial of (4.13) differs from that given by (4.19), we have $\beta_\lambda(P(y_1, ..., y_q)) = 0$ is a desired relation, and then set $\varpi = u'$ where u' is resolved from (4.14).

Finally, we say that an odd dimensional element $\varpi \in V$ is *associated* with (4.13) if ϖ is given by one of items (i)–(iii) above. In particular, ϖ always exists for $\widetilde{H}_{\mathbb{Q}} = 0$ or, more generally, for $z \in \mathcal{H}_{\Bbbk}$.

Given an even dimensional $y \in \mathcal{H}_{\Bbbk,0}^{ev}$ with the relation

$$P(y) = \lambda_y y^m = 0, \quad m \geq 2,\ \lambda_y \in \mathbb{Z}, \tag{4.20}$$

it rises to the sequence $\{y_n \in V\}_{n\geq 0}$ in (RH,d) :

$$\begin{aligned} dy_{2k+1} &= \sum_{\substack{i+j=k-1\\ i,j\geq 0}} y_{2i+1}y_{2j+1} - \sum_{\substack{i_1+\cdots+i_m=k\\ 0\leq i_1\leq\ldots\leq i_m\leq k}} \lambda_y y_{2i_1}\cdots y_{2i_m}, \\ dy_{2k} &= \sum_{\substack{i+j=2k-1\\ i,j\geq 0}} (-1)^{i+1}y_iy_j, \qquad k\geq 0, \end{aligned} \tag{4.21}$$

where y_n is of odd degree for $n=2k+1$ and is of even degree for $n=2k$. In fact a straightforward check shows that each $y_{2k}, k\geq 1$, can be expressed in terms of y_r for $r<2k$ as $y_{2k}=-y_0\smile_1 y_{2k-1}$ mod $\mathcal{D}$ (e.g. $y_2=-y_0\smile_1 y_1+\lambda_z \sum_{i+j=n-1} y_0^i(y_0\cup_2 y_0)y_0^j$). Consequently, $h^{tr}(y_{2k})\in\mathcal{D}$.

Proposition 5. *Let $x\in\mathcal{H}^{od}_{\Bbbk}$ and let (4.20) be a single relation in $H_{\Bbbk}$ with $|P|<|x|$. Then $x\notin \operatorname{Ker}\sigma$ and if there is $b\in H_{\Bbbk,1}$ with $\beta_\lambda(b)=x$, then also $b\notin\operatorname{Ker}\sigma$.*

Proof. Theoretically there may be $y_n\in V$ given by (4.21) serving as a source for h to kill x or b. Since $|x|$ is odd and $|h^{tr}y_{2k+1}|$ is even (and $h^{tr}(y_{2k})\in\mathcal{D}$), x_0 is not λ-homologous to zero. When there is $b\in H_{\Bbbk,1}$ with $\beta_\lambda(b)=x$, the element $b_0\in K_\mu$ is not λ-homologous to zero since (4.21) and $d_h^2=0$ prevent b_0 to be in the target of h evaluated on any y_{2k+1}. □

Proposition 6. *Let $x\in\mathcal{H}^{od}_{\Bbbk}$ and let $P(y_1,...,y_q)=0$ be a relation given by (4.13).*

(i) If $|x|\leq|P|$ or (4.13) is specified as (4.20) to be a single relation with $|P|<|x|$, then the sequence $\mathbf{x}=\{x(i)\}_{i\geq 0}$ given by (4.5) for $w=x_0$ satisfies (4.1)–(4.2) in $RH\otimes\bar V$.

(ii) Let $\varpi\in V$ be associated with the relation $P(y_1,...,y_q)=0$.

(ii_1) If $P(y_1,...y_q)$ is of the smallest degree with $|x|<|P|$, then the sequence $\mathbf{x}=\{x(i)\}_{i\geq 0}$ given by (4.5) for $w=\varpi$ satisfies (4.1)–(4.2) in $RH\otimes\bar V$.

(ii_2) Let $P'(y'_1,...,y'_{q'})=0$ and $P(y_1,...y_q)=0$ be two relations of the smallest degree with $|x|<|P'|\leq|P|$ where the first relation is given by (4.18). Then the sequence $\mathbf{x}=\{x(i)\}_{i\geq 0}$ given by (4.5) for $w=\varpi$ satisfies (4.1)–(4.2) in $RH\otimes\bar V$.

(iii) Let $P(y_1,...,y_q)\neq\lambda_x x^2$ for μ even and $\mathcal{P}_1(x)=0$, $\lambda_x\in\mathbb{Z}$. Then the pair of sequences given by items (i) and (ii) is admissible.

Proof. (i) First note that when $|x|\leq|P|$, x_0 is not λ-homologous to zero by the degree reason, while apply Proposition 5 for $|P|<|x|$; the same argument implies that b_0 is also not λ-homologous to zero when $\beta_\lambda(b)=x$. Consider two subcases.

(i_a) Let $x\in H_{\Bbbk,0}$. In fact we have to verify only (4.1). (i_{a1}) Assume there is no b with $\beta_\lambda(b)=x$. Observe that for an odd dimensional $a\in RH$ and $n\geq 2$, $da^{\smile_1 n}$ contains a summand component of the form

$$-\sum_{k+\ell=n}\binom{n}{k}a^{\smile_1 k}a^{\smile_1\ell},\quad k,\ell\geq 1\quad(\text{with}\ \ a^{\smile_1 1}=a). \tag{4.22}$$

By setting $v={x_0}^{\smile_1 n}$ and $v_1v_2=-\binom{n}{k}{x_0}^{\smile_1 k}{x_0}^{\smile_1\ell}$, some k,ℓ, we see that the hypothesis of Proposition 3 is satisfied and hence ${x_0}^{\smile_1 n}$ is not weakly homologous to zero. Consequently, $[x(n)]\neq 0$ as desired. (i_{a2}) Assume there is b with $\beta_\lambda(b)=x$.

Let p be a prime that divides μ. We have a sequence of relations in (RH, d)

$$db_n = \sum_{\substack{i+j=n\\ i\geq 0; j\geq 1}} \varepsilon_n \binom{n+1}{i+1} b_i x_0{}^{\smile_1 j} + \varepsilon_n \lambda\, x_0{}^{\smile_1 (n+1)}, \quad b_n \in V,\ n \geq 1, \tag{4.23}$$

where $\varepsilon_{p^k-1} = \frac{1}{p}$ and $\varepsilon_n = 1$ for $n+1 \neq p^k, k \geq 1$. In view of (4.23) and Proposition 3 we remark that $x_0{}^{\smile_1 n}$ may be weak homologous to zero only for $\lambda = \mu = p = n+1$. In any case consider the element $a_n \in (RH, d_h)$ given by

$$a_n = \sum_{\substack{i+j=n\\ i\geq 0; j\geq 1}} \varepsilon_n \binom{n+1}{i+1} b_i x_0{}^{\smile_1 j} + hb_n.$$

Since b_0 and x_0 are not λ-homologous to zero, so is b_n for all n. Obviously $d_h a_n \in \widetilde{\mathcal{D}}$ and by setting $a = a_n$ and $v_1 \cdot v_2 = \varepsilon_n (n+1)\, b_{n-1} \cdot x_0$, the component of a for $(i, j) = (n-1, 1)$, the hypotheses of Proposition 4 are satisfied. Therefore, we get $[\chi_1(a)] \neq 0$ in $H(RH \otimes \bar{V}, d_\omega)$. Obviously $[\chi_1(a)] = -\varepsilon_n \lambda [\chi_1(x_0{}^{\smile_1 (n+1)})]$. Thus $[x(n)] \neq 0$ as desired.

(i_b) Let $x \in H_{\Bbbk,1}$. Then we have to verify only (4.2). Since $x_0 \in K_\mu$ is not λ-homologous to zero, the proof easily follows from the analysis of the component given by (4.22) for $a = x_0$ in $dx_0{}^{\smile_1 n}$.

(ii) When ϖ is not λ-homologous to zero and either $d_h\varpi \in \widetilde{\mathcal{D}}_\Bbbk$ or $d_h\varpi \notin \widetilde{\mathcal{D}}_\Bbbk$ but $z_j{}^{\smile_1 (p^{\nu_j+1}-1)}$ is also not λ-homologous to zero for all j (z_j is a variable in $d_h\varpi$), the proof is analogous to that of subcase (i_{a1}) or (i_b) of item (i). Otherwise, we observe that ϖ is again not λ-homologous to zero in (ii_1), while ϖ may be λ-homologous to zero in (ii_2) only by evaluating h on certain elements $u_i \in V$ arising from the relation given by (4.18) the first of which is $u = u_0$ as given by (4.14); since $du \notin \widetilde{\mathcal{D}}_\Bbbk$, neither du_i is in $\widetilde{\mathcal{D}}_\Bbbk$. And then in the both subcases a straightforward check completes the proof.

(iii) The proof is analogous to that of items (i)–(ii). The restriction on the relation for μ even is in fact explained by Example 1.

□

Remark 4. *Let $A_\Bbbk = C^*(X; \mathbb{Z}_p), p > 2$. The case of $x \in H_{\mathbb{Z}_p}$ with $\beta(b) = x$ fundamentally distinguishes the (based) loop and free loop spaces on X with respect to the existence of infinite sequence arising from x in $H^*(\Omega X; \mathbb{Z}_p)$ and $H^*(\Lambda X; \mathbb{Z}_p)$ respectively. Namely, let both $\mathcal{P}_1(x)$ and $\langle x\rangle^p$ be multiplicative generators of $H^*(X; \mathbb{Z}_p)$ such that $\mathcal{P}_1(x) \in \langle b, x, ..., x\rangle$, the p^{th}-order Massey product. Then by the hypotheses of Proposition 6 the sequence arising from x in $H^*(\Omega X; \mathbb{Z}_p)$ may terminate at the p^{th}-component* (*see* [16] *for $p = 3$*), *while is always infinite in $H^*(\Lambda X; \mathbb{Z}_p)$.*

5. Proof of Theorem 2

The proof of the theorem relies on the two basic propositions below in which the condition that $\tilde{H}_\Bbbk$ requires at least two algebra generators is treated in two specific cases. Note also that the essence of the method used in the proof of the following proposition is in fact kept for μ to be a prime.

Proposition 7. *Let $H_\Bbbk$ be a finitely generated $\Bbbk$-module with $\mu \geq 2$. If $\tilde{H}_\Bbbk$ requires at least two algebra generators and $\tilde{H}_\mathbb{Q}$ is either trivial or has a single algebra generator, then there are two sequences $\mathbf{x}_\mu = \{x_\mu(i)\}_{i\geq 0}$ and $\mathbf{y}_\mu = \{y_\mu(j)\}_{j\geq 0}$ of*

mod μ d_ω-*cocycles in* $(RH\otimes\bar{V},d_\omega)$ *whose degrees form arithmetic progressions and the product classes* $\{[t_{\Bbbk}x_\mu(i)]\cdot[t_{\Bbbk}y_\mu(j)]\}_{i,j\geq 0}$ *are linearly independent in* $H(RH\otimes\bar{V}_{\Bbbk},d_\omega)$.

Proof. First, we exhibit two sequences $\mathbf{x}=\{x(i)\}_{i\geq 0}$ and $\mathbf{y}'=\{y'(i)\}_{i\geq 0}$ in $(RH\otimes\bar{V},d_\omega)$ consisting of mod μ d_ω-cocycles and satisfying the hypotheses of Proposition 6. In the case $\tilde{H}_{\mathbb{Q}}\neq 0$, let $\mathfrak{z}$ be a single multiplicative generator of infinite order of H and $z=t^*_{\Bbbk}(\mathfrak{z})\in H_{\Bbbk,0}$.

By the hypotheses of the proposition there is an odd dimensional element $x\in\mathcal{H}_{\Bbbk}$ and choose x to be of the smallest degree. Define $\mathbf{x}=\{x(i)\}_{i\geq 0}$ by (4.5) for $w=x_0$.

To find the second sequence, note that there must be an even dimensional element $y\in\mathcal{H}_{\Bbbk}$ and hence a relation $y^{\hbar_y+1}=0$ in $H_{\Bbbk}$ unless maybe μ is even and $x=z$ with $\mathcal{P}_1(x)\neq 0$ in which case we have a relation $x^{\hbar_x+1}=0$ instead. First, observe the following: if there is $y\in\mathcal{H}^{od}_{\Bbbk}$ with $y\notin\operatorname{Ker}\sigma$ and linearly independent with $\mathcal{P}^{(m)}_1(x)$ for some m, define $\mathbf{y}'=\{y'(j)\}_{j\geq 0}$ by (4.5) for $w=y_0$; If there is $y\in\mathcal{H}^{od}_{\Bbbk,0}$ linearly dependent on $\mathcal{P}^{(m)}_1(x)$, while $\mathcal{P}^{(m-1)}_1(x)\notin\operatorname{Ker}\sigma$, define $\mathbf{y}'=\{y'(j)\}_{j\geq 0}$ by (4.5) for w given by (4.11) in which x is replaced by $\mathcal{P}^{(m-1)}_1(x)$. Otherwise, consider a relation $P(y_1,...,y_q)=0$ of the smallest degree in $H_{\Bbbk}$ unless $P(y_1,...,y_q)=\lambda_x x^2,\lambda_x\in\mathbb{Z}$, whenever μ is odd or μ is even and $\mathcal{P}_1(x)=0$. When the relation admits to associate ϖ as in subsection 4.3, define $\mathbf{y}'=\{y'(j)\}_{j\geq 0}$ by (4.5) for $w=\varpi$. When the definition of ϖ is obstructed, consider the next relation in $H_{\Bbbk}$. This time the second relation admits to associate ϖ (since the above $\mathfrak{z}$ is unique) and hence the second sequence $\mathbf{y}'=\{y'(j)\}_{j\geq 0}$ is defined.

Finally, the pair of the sequences $(\mathbf{x},\mathbf{y}')$ found above is admissible: when the existence of the pair involves relation(s) in $H_{\Bbbk}$ (if not, the claim is rather obvious), it satisfies the hypotheses of Proposition 6. Obtain the associated sequences $\mathbf{x}_\mu=\{x_\mu(i)\}_{i\geq 0}$ and $\mathbf{y}_\mu=\{y_\mu(j)\}_{j\geq 0}$ as in subsection 4.1. The explicit product on $RH\otimes\bar{V}$ allows us to ensure immediately that the product classes $\{[t_{\Bbbk}x_\mu(i)]\cdot[t_{\Bbbk}y_\mu(j)]\}_{i,j\geq 0}$ are linearly independent in $H(RH\otimes\bar{V}_{\Bbbk},d_\omega)$. □

Given a cochain complex (C^*,d) over $\mathbb{Q}$, let $S_C(T)=\sum_{n\geq 0}(\dim_{\mathbb{Q}}C^n)T^n$ and $S_{H(C)}(T)=\sum_{n\geq 0}(\dim_{\mathbb{Q}}H^n(C))T^n$ be the Poincaré series. Recall the convention: $\sum_{n\geq 0}a_nT^n\leq\sum_{n\geq 0}b_nT^n$ if and only if $a_n\leq b_n$. The following proposition can be thought of as a modification of Proposition 3 in [20] for the non-commutative case.

Proposition 8. *Let* (B^*,d_B) *be a dga over* $\mathbb{Q}$ *and let* $y\in B^k,k\geq 2$, *be an element such that* $d_By=0$ *and* $yb\neq 0$ *for all* $b\in B$. *Then*

$$S_{H(B/yB)}(T)\leq(1+T^{k-1})S_{H(B)}(T). \tag{5.1}$$

Proof. We have an inclusion of cochain complexes $s^kB\xrightarrow{\iota}B$ induced by the map $B\xrightarrow{y\cdot}B$, $b\to yb$, and, consequently, the short exact sequence of cochain complexes

$$0\longrightarrow s^kB\xrightarrow{\iota}B\longrightarrow B/yB\longrightarrow 0.$$

Then the proof of the proposition is entirely analogous to that of [17, Proposition 7]. □

Proposition 9. *Let* $H_{\mathbb{Q}}$ *be a finitely generated* $\mathbb{Q}$*-module. If* $\tilde{H}_{\mathbb{Q}}$ *requires at least two algebra generators, then the sequence* $\{\varsigma_i(A)\}$ *grows unbounded.*

Proof. Denote $(B, d_B) = (RH\otimes\bar{V}_{\mathbb{Q}}, d_\omega)$. We will define two sequences $\mathbf{x} = \{x(i)\}_{i\geq 0}$ and $\mathbf{y} = \{y(j)\}_{j\geq 0}$ in B consisting of d_B-cocycles in the four cases below. Consider two relations of the smallest degree in $H_{\mathbb{Q}}$

$$P_1(x_1, ..., x_p) = 0 \quad\text{and}\quad P_2(y_1, ..., y_q) = 0.$$

Suppose that

(i) All x_i and y_j are even dimensional. Obtain u_1 and u_2 from (4.14) that correspond to the above relations, and define $\mathbf{x} = \{x(i)\}_{i\geq 0}$ and $\mathbf{y} = \{y(j)\}_{j\geq 0}$ by (4.5) for $w = u_1$ and $w = u_2$ respectively.

(ii) There are odd dimensional elements c_1 and c_2 among x_i's and y_j's respectively. Then define $\mathbf{x} = \{x(i)\}_{i\geq 0}$ and $\mathbf{y} = \{y(j)\}_{j\geq 0}$ by

$$x(i) = 1\otimes s^{-1}\left(c_1{}^{\smile_1 (i+1)}\right) \tag{5.2}$$

and

$$y(j) = 1\otimes s^{-1}\left(c_2{}^{\smile_1 (j+1)}\right) \tag{5.3}$$

respectively.

(iii) There is a single odd dimensional x_i and all y_j are even dimensional. Define $\mathbf{x} = \{x(i)\}_{i\geq 0}$ by (5.2), while define $\mathbf{y} = \{y(j)\}_{j\geq 0}$ as in item (i). When all x_i are even dimensional and a single y_j is odd dimensional, define $\mathbf{x} = \{x(i)\}_{i\geq 0}$ as in item (i), while define $\mathbf{y} = \{y(j)\}_{j\geq 0}$ by (5.3).

(iv) There is a single odd dimensional x_i and a single odd dimensional y_j equal to the same element $a\in H_{\mathbb{Q}}$. Then obviously $P_1(x_1, ..., x_p)$ admits a representation $P_1(x_1, ..., x_p) = ab$ for a certain even dimensional element $b\in H_{\mathbb{Q}}$. Consequently, the corresponding relation in $(RH_{\mathbb{Q}}, d)$ given by (4.14) has the form $du = a_0b_0$ for $a_0 \in \mathcal{V}^{0,*}$ and $b_0\in R^0H^*$. Denote $a_1 = u$ and $b_1 = -a_1 - a_0 \smile_1 b_0$ to obtain $db_1 = -b_0a_0$. Furthermore, denoting $b_2 = -b_0 \smile_1 a_1$, there are the induced relations in $(RH_{\mathbb{Q}}, d)$:

$$\begin{array}{lll} da_2 = a_0b_1 - a_1a_0, & da_3 = a_0b_2 - a_1a_1 - a_2b_0, & \\ db_2 = -b_0a_1 - b_1b_0, & db_3 = -b_0a_2 - b_1b_1 + b_2a_0, & a_3, b_3\in V_{\mathbb{Q}}. \end{array}$$

Thus we have $h(a_3 - b_3) = (h^2 + h^3)(a_3 - b_3)$ with $h^2(a_3 - b_3) = -b_0 \smile_1 h^2a_2$ in $(RH_{\mathbb{Q}}, d_h)$. Let $b_0 = P'(z_1, ..., z_r)$ and $h^3(a_3 - b_3) = P''(z_{r+1}, ..., z_m)$ for some $z_j\in V_{\mathbb{Q}}^{0,*}$, $1\leq j\leq m$, $1\leq r < m$. Define a complex (D, d_D) as $(D, d_D) = (B/1\otimes\bar{C}, d_D)$, where $\bar{C}\subset\bar{V}_{\mathbb{Q}}$ is a subcomplex (additively) generated by the expressions

$$\{\bar{z}_j\,, \overline{z_j\smile_1 v}\mid v\in V_{\mathbb{Q}},\, 1\leq j\leq m\}.$$

Define $\bar{x}$ and $\bar{y}$ in $\bar{V}_{\mathbb{Q}}/\bar{C}$ as the projections of the elements $\bar{a}_0$ and $\overline{a_3 - b_3}$ under the quotient map $\bar{V}_{\mathbb{Q}}\to\bar{V}_{\mathbb{Q}}/\bar{C}$ respectively. Then $1\otimes\bar{x}$ and $1\otimes\bar{y}$ are cocycles in (D, d_D). Apply formulas (5.2)–(5.3) for $(c_1, c_2) = (x, y)$ to obtain the sequences $\mathbf{x} = \{x(i)\}_{i\geq 0}$ and $\mathbf{y} = \{y(j)\}_{j\geq 0}$ in D. Then the product classes $\{[x(i)]\cdot[y(j)]\}_{i,j\geq 0}$ are linearly independent in $H(D\,, d_D)$. Finally, apply Proposition 8 successively for $y\in\{z_1, ..., z_m\}$ to obtain $S_{H(D)}(T)\leq S_{H(B)}(T)$, and then an application of Proposition 1 completes the proof. $\square$

5.1. **Proof of Theorem 2.** In view of Proposition 1, the proof reduces to the examination of the $\Bbbk$-module $H(RH\otimes \bar{V}_{\Bbbk}, d_{\omega})$. If $\tilde{H}_{\Bbbk}$ has a single algebra generator a, then the set $\{\varsigma_i(A)\}$ is bounded since $\varsigma_i(A)=2$ (cf. [6]). If $\tilde{H}_{\Bbbk}$ requires at least two algebra generators, then the proof follows from Proposition 1 and Proposition 7 for $\mu\geq 2$, and from Proposition 9 for $\mu=0$.

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